1) Can write $X(s,v) = (\cos(s) - v\sin(s), \sin(s) + v\cos(s), v)$

Notice that

$$(\cos(s) - v\sin(s))^2 + (\sin(s) + v\cos(s))^2 - v^2$$

$$= \cos^2(s) - 2v\cos(s)\sin(s) + v^2\sin^2(s) + \sin^2(s) + 2v\cos(s)\sin(s) + v^2\cos^2(s) - v^2$$

$$= 1 + v^2 - v^2 = 1$$

So $X(s,v)$ is part of the hyperboloid $x^2+y^2-z^2=1$.

X is a surjective map on the hyperboloid: let (x,y,z) satisfy $x^2+y^2-z^2=1$. Then take $v=z \Rightarrow x^2+y^2 = 1+z^2 = 1+v^2$.

Rewriting in polar coordinates, we have $x = \sqrt{1+v^2}\cos\theta$

$y = \sqrt{1+v^2}\sin\theta$ for some θ.

$$\begin{cases} \cos(s) - v\sin(s) = \sqrt{1+v^2}\cos\theta & (1) \\ \sin(s) + v\cos(s) = \sqrt{1+v^2}\sin\theta & (2) \\ v = z \end{cases}$$

By considering $(1)\sin(s) - (2)\cos(s)$ and $(1)\cos(s) + (2)\sin(s)$, the above system of equations can be solved by setting

$v = z$

$\sin(\theta - s_0) = \frac{v}{\sqrt{1+v^2}}$ for some $s_0 \in [0, 2\pi]$ which exists.

$\cos(\theta - s_0) = \frac{1}{\sqrt{1+v^2}}$

Then $X(s_0, v) = (x,y,z)$ on Hyperboloid. So X is surjective.

X is not injective since $X(0,v) = (1, v, v) = X(2\pi, v)$.

However if you exclude one of the end-points $0, 2\pi$, then X is injective

X is full rank on $0<s<2\pi$ since :

$$X_s = (-\sin(s) - v\cos(s),\ \cos(s) - v\sin(s),\ 0)$$

$$X_v = (-\sin(s),\ \cos(s),\ 1)$$

$$X_s \times X_v = (\cos(s) - v\sin(s),\ v\cos(s) + \sin(s),\ -v)$$

$$|X_s \times X_v| = (\cos(s) - v\sin(s))^2 + (v\cos(s) + \sin(s))^2 + (-v)^2$$

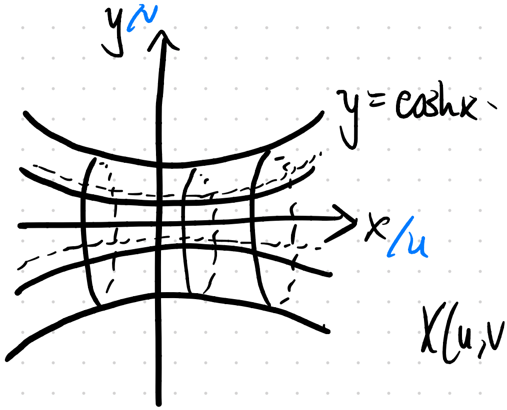
$$= \cos^2(s) - 2v\sin(s)\cos(s) + v^2\sin^2(s)$$

$$+ v^2\cos^2(s) + 2v\cos(s)\sin(s) + \sin^2(s) + v^2$$

$$= 1 + v^2(\cos^2(s) + \sin^2(s)) + v^2$$

$$= 1 + 2v^2 \geq 1 > 0.$$

2)



So let $\alpha(u)$ be a curve given by

$$\alpha(u) = (u, \cosh u)$$

Then revolving around the x-axis is given by

$$X(u,v) = (u, \alpha(u)\cos v, \alpha(u)\sin v)$$
$$= (u, \cosh u \cos v, \cosh u \sin v)$$

$u \in \mathbb{R}$, $0 \leq v < 2\pi$.

$$X_u = (1, \sinh u \cos v, \sinh u \sin v)$$

$$X_v = (0, -\cosh u \sin v, \cosh u \cos v)$$

$$E = \langle X_u, X_u \rangle = 1 + \sinh^2 u = \cosh^2 u.$$

$$F = \langle X_u, X_v \rangle = 0$$

$$G = \langle X_v, X_v \rangle = \cosh^2 u.$$

3) · Clearly X is smooth.

- Check linear independence in the columns of dX:

$$X_u = (1-u^2+v^2,\ 2uv,\ 2u)$$

$$X_v = (2uv,\ 1-v^2+u^2,\ -2v)$$

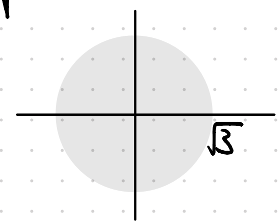
alternatively, can also consider $0 = aX_u + bX_v$ and show a, b must be 0.

$$X_u \times X_v = (-2u-2u^3-2uv^2,\ 2v+2u^2v+2v^3,\ 1-u^4-2u^2v^2-v^4)$$
$$= (-2u(1+u^2+v^2),\ 2v(1+u^2+v^2),\ 1-(u^2+v^2)^2)$$

$$|X_u \times X_v|^2 = (-2u-2u^3-2uv^2)^2 + (2v+2u^2v+2v^3)^2 + (1-u^4-2u^2v^2-v^4)^2$$
$$= (1+u^2+v^2)^4 \geq (1+0+0)^4 = 1 > 0.$$

So X_u, X_v are linearly independent and dX has full rank.

- Check homeomorphism in the domain $\{u^2+v^2<3\}$.



Suppose for contradiction that $(u_1, v_1) \neq (u_2, v_2)$ s.t. $X(u_1,v_1) = X(u_2,v_2)$. Then we have that

(1) $u_1 - \frac{u_1^3}{3} + u_1v_1^2 = u_2 - \frac{u_2^3}{3} + u_2v_2^2$, and

(2) $u_1^2 - v_1^2 = u_2^2 - v_2^2$.

$$(1) \iff 0 = u_2 - u_1 - \tfrac{1}{3}(u_2^3 - u_1^3) + u_2v_2^2 - u_1v_1^2$$
$$= u_2 - u_1 - \tfrac{1}{3}(u_2-u_1)(u_2^2+u_2u_1+u_1^2) + u_2v_2^2 - u_1v_1^2$$
$$= u_2 - u_1 - \tfrac{1}{3}(u_2-u_1)(u_2^2+u_2u_1+u_1^2) + (u_2-u_1)v_2^2 + u_1(v_2^2-v_1^2)$$
$$= u_2 - u_1 - \tfrac{1}{3}(u_2-u_1)(u_2^2+u_2u_1+u_1^2) + (u_2-u_1)v_2^2 + u_1(u_2^2-u_1^2) \quad \leftarrow \text{by (2)}$$
$$= (u_2-u_1)\left(1 - \tfrac{1}{3}(u_2^2+u_2u_1+u_1^2) + v_2^2 + u_1(u_2+u_1)\right)$$

Since we are supposing $u_2 \neq u_1$, we must have

$$0 = 1 - \frac{1}{3}(u_2^2 + u_2u_1 + u_1^2) \cancel{+v_2^2} + u_1(u_2+u_1)$$

$$\geq 1 - \frac{1}{3}u_2^2 + \frac{2}{3}u_2u_1 + \frac{2}{3}u_1^2$$

$$= 1 + \frac{1}{3}(u_1^2 + 2u_1u_2 + u_2^2) - \frac{2}{3}u_2^2 + \frac{1}{3}u_1^2$$

$$= 1 + \cancel{\frac{1}{3}(u_1+u_2)^2} - \frac{2}{3}u_2^2 + \frac{1}{3}u_1^2$$

$$\geq 1 - \frac{2}{3}u_2^2 + \frac{1}{3}u_1^2$$

$$= 1 + \frac{1}{3}(u_1^2 - u_2^2) - \frac{1}{3}u_2^2$$

$$= 1 + \frac{1}{3}(v_1^2 - v_2^2) - \frac{1}{3}u_2^2$$

$$= 1 - \frac{1}{3}(u_2^2 + v_2^2) + \cancel{\frac{1}{3}v_1^2}$$

$$\geq 1 - \frac{1}{3}(u_2^2 + v_2^2) > 0 \quad \text{by } u_2^2 + v_2^2 < 3.$$

a contradiction since the conclusion here is $0 > 0$.

Hence X is injective in the domain $\{u^2+v^2<3\}$. X is continuous and X is surjective onto its image. So X^{-1} exists locally. Furthermore, since dX is nonsingular, by Inverse Function Theorem X^{-1} is smooth. So homeomorphism condition is satisfied. (Alternatively, can also check X is an open map).

So X is a regular surface patch

- Consider $X((\sqrt{3},0)) = (\sqrt{3} - \frac{3\sqrt{3}}{3}, 0, 3) = (0,3)$

$$X((-\sqrt{3},0)) = \left(-\sqrt{3}+\frac{3\sqrt{3}}{3}, 0, 3\right) = (0,3).$$

So $(\sqrt{3},0)$, $(-\sqrt{3},0)$ are two such points.

- Recall

$$X_u = (1-u^2+v^2, 2uv, 2u)$$

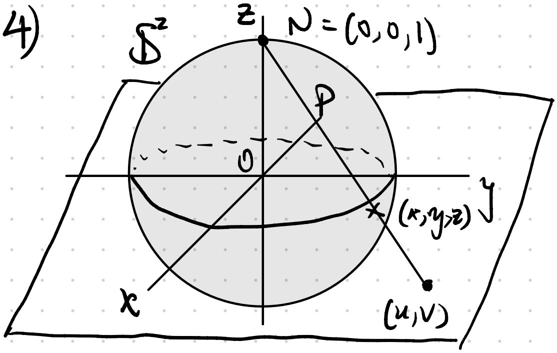
$$X_v = (2uv, 1-v^2+u^2, -2v)$$

$$X_u \times X_v = \left(-2u(1+u^2+v^2), 2v(1+u^2+v^2), 1-(u^2+v^2)^2\right)$$

$$|X_u \times X_v| = (1+u^2+v^2)^2.$$

So $N = \dfrac{X_u \times X_v}{|X_u \times X_v|} = \left(\dfrac{-2u}{1+u^2+v^2}, \dfrac{2v}{1+u^2+v^2}, \dfrac{1-u^2-v^2}{1+u^2+v^2}\right)$ //.

4)



Let $(u,v)\in\mathbb{R}^2$, P be the line connecting (u,v) to $N=(0,0,1)$. Note: we are projecting onto the $z=0$ plane, so (u,v) has coords. in $\mathbb{R}^3$ $(u,v,0)$. Parametrize P by (also okay if you projected onto $z=-1$ plane).

$$P(t)=(0,0,1)+t(u,v,-1)$$

so that at $t=1$, $P(1)=(u,v,0)$.

Solve for t s.t. $P(t)$ lies on the sphere S^2. At S^2, $|P(t)|^2=1$, i.e.

$$(tu)^2+(tv)^2+(1-t)^2=1$$

$$t^2u^2+t^2v^2+\cancel{1}-2t+t^2=\cancel{1} \Rightarrow 2=t(1+u^2+v^2).$$

$$\Rightarrow t=\frac{2}{1+u^2+v^2}.$$

So if $t=\frac{2}{1+u^2+v^2}$, then $\left(\frac{2u}{1+u^2+v^2},\frac{2v}{1+u^2+v^2},\frac{-1+u^2+v^2}{1+u^2+v^2}\right)\in S^2$ by construction.

So $X(u,v)=\left(\frac{2u}{1+u^2+v^2},\frac{2v}{1+u^2+v^2},\frac{-1+u^2+v^2}{1+u^2+v^2}\right)$.

You can check that the 3 conditions for a regular surface are satisfied by X.

$$X_u=\left(\frac{-2u^2+2v^2+2}{(1+u^2+v^2)^2},\frac{-4uv}{(1+u^2+v^2)^2},\frac{4u}{(1+u^2+v^2)^2}\right)$$

$$X_v=\left(\frac{-4uv}{(1+u^2+v^2)^2},\frac{2u^2-2v^2+2}{(1+u^2+v^2)^2},\frac{4v}{(1+u^2+v^2)^2}\right)$$

$$E=\langle X_u,X_u\rangle=\left(\frac{-2u^2+2v^2+2}{(1+u^2+v^2)^2}\right)^2+\left(\frac{-4uv}{(1+u^2+v^2)^2}\right)^2+\left(\frac{4u}{(1+u^2+v^2)^2}\right)^2$$

$$=\left(\frac{1}{1+u^2+v^2}\right)^4\left((-2u^2+2v^2+2)^2+(-4uv)^2+(4u)^2\right)$$

$$= \left(\frac{1}{1+u^2+v^2}\right)^4 \left(4u^4 + 8u^2v^2 + 8u^2 + 4v^2 + 8v^2 + 4\right)$$

$$= \frac{4}{(1+u^2+v^2)^2}.$$

$$F = \langle X_u, X_v \rangle = \left(\frac{1}{1+u^2+v^2}\right)^4 \left(-4uv\left(-2u^2+2v^2+2\right) - 4uv\left(2u^2-2v^2+2\right) + 16uv\right)$$

$$= \left(\frac{1}{1+u^2+v^2}\right)^4 \left(\cancel{8u^3v} - \cancel{8uv^3} - \cancel{8uv} - \cancel{8u^3v} + \cancel{8uv^3} - \cancel{8uv} + \cancel{16uv}\right)$$

$$= 0.$$

$$G = \langle X_v, X_v \rangle = \left(\frac{1}{1+u^2+v^2}\right)^4 \left((-4uv)^2 + (2u^2-2v^2+2)^2 + (4v)^2\right)$$

$$= \frac{4}{(1+u^2+v^2)^2}. \quad //.$$

5) $X_u = (-\sin v \sin u, \sin v \cos u, 0)$

$X_v = (\cos v \cos u, \cos v \sin u, -\sin v)$

$E = \langle X_u, X_u \rangle = (-\sin v \sin u)^2 + (\sin v \cos u)^2$
$= \sin^2 v \sin^2 u + \sin^2 v \cos^2 u$
$= \sin^2 v.$

$F = \langle X_u, X_v \rangle = -\sin v \sin u \cos v \cos u + \sin v \cos u \cos v \sin u = 0$

$G = \langle X_v, X_v \rangle = (\cos v \cos u)^2 + (\cos v \sin u)^2 + (-\sin v)^2$
$= \cos^2 v \cos^2 u + \cos^2 v \sin^2 u + \sin^2 v$
$= \cos^2 v (\cos^2 u + \sin^2 u) + \sin^2 v$
$= \cos^2 v + \sin^2 v = 1.$

$\alpha(t) = (\sin t \cos u_0, \sin t \sin u_0, \cos t)$ $u(t) = u_0$, $v(t) = t$ $(a \leq t \leq b)$

$\frac{du}{dt} = 0$, $\frac{dv}{dt} = 1.$

$l(\alpha) = \int_a^b (\sin^2 v \cdot 0^2 + 2 \cdot 0 \cdot 0 \cdot 1 + 1 \cdot 1^2)^{1/2} dt = \int_a^b dt = b - a.$

Since this is the distance of a straight line between two points,

$l(\beta) = \int_a^b |\beta'(t)| dt = \int_a^b \sqrt{\sin^2 v \, u'(t)^2 + v'(t)^2} dt \geq \int_a^b |v'(t)| dt$

$\geq \int_a^b v'(t) dt = v(b) - v(a) = b - a = l(\alpha).$

Since you can show $v(a) = a$, $v(b) = b$.

6) Torus parametrized by

$$X(u,v) = ((a+r\cos u)\cos v,\ (a+r\cos u)\sin v,\ r\sin u) \quad 0<u,v<2\pi$$

$$X_u = (-r\cos v\sin u,\ -r\sin v\sin u,\ r\cos u)$$

$$X_v = (-(a+r\cos u)\sin v,\ (a+r\cos u)\cos v,\ 0)$$

$$\begin{aligned} E = \langle X_u, X_u\rangle &= (-r\cos v\sin u)^2 + (-r\sin v\sin u)^2 + (r\cos u)^2 \\ &= r^2\cos^2 v\sin^2 u + r^2\sin^2 v\sin^2 u + r^2\cos^2 u \\ &= r^2(\sin^2 u(\cos^2 v + \sin^2 v) + \cos^2 u) \\ &= r^2 \end{aligned}$$

$$F = \langle X_u, X_v\rangle = 0$$

$$\begin{aligned} G = \langle X_v, X_v\rangle &= (-(a+r\cos u)\sin v)^2 + ((a+r\cos u)\cos v)^2 \\ &= (a+r\cos u)^2\sin^2 v + (a+r\cos u)^2\cos^2 v \\ &= (a+r\cos u)^2 \end{aligned}$$

So $\sqrt{EG-F^2} = \sqrt{r^2(a+r\cos u)^2} = r(r\cos u + a)$.

$$A(R) = \int_0^{2\pi}\int_0^{2\pi} r^2\cos u + ra \, dv\, du = \int_0^{2\pi}(r^2\cos u + ra)\,du \int_0^{2\pi} dv$$

$$= 2\pi\int_0^{2\pi}(r^2\cos u + ra)\,du = 2\pi\left(r^2\sin u + rau\right)\Big|_0^{2\pi}$$

$$= 4\pi^2 ra$$